SOCIAL CLASSES AND EQUILIBRIUM*

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Abstract

This essay develops a simple worker-capitalist model. We first show that if labor markets clear at low wages then certain produced commodities are over-supplied due to inadequate demand. This implies that any voluntary trade between firms and consumers is Pareto-inefficient. Nonetheless, this inefficiency can be rectified by a Pareto-improving income transfer from capitalists to workers. We claim that these results point out an efficiency enhancing role for welfare state, and demonstrate markets' incompetency for coordinating economic activity efficiently evoking the Marxian/Keynesian polemics. Finally, we analyze price cycles in a simple setting of Walrasian price adjustment.

Keywords: Disequilibrium, underconsumption, Arrow's corner, irreducibility, social classes.
JEL classification: C62 ; D31; D50

SOSYAL SINIFLAR VE DENGE

Özet

Bu makalede basit bir işçi-kapitalist modeli oluşturulmuştur. İlk olarak emek piyasaları düşük ücretlerde dengelenirse yetersiz talep dolayısıyla belli üretilmiş malların aşırı üretildiğini gösteriyoruz. Bu sonuç firmalar ve tüketiciler arasındaki tüm gönüllü ticari işlemlerin Pareto-verimsiz olduğu anlamına gelmektedir. Ancak bu verimsizlik sorunu kapitalistlerden işçilere doğru Pareto-iyileştirici gelir transferi aracılığı ile çözülebilir. Bu sonuçların refah devleti adına etkinlik arttırıcı bir rolü işaret ettiğini ve Marksist/Keynesci tartışmaları hatırlatır şekilde piyasaların iktisadi faaliyeti etkin şekilde koordine edemediğini gösterdiğini savunuyoruz. Son olarak basit bir Walrasgil kurguda fiyat dalgalanmalarını inceliyoruz.

Anahtar kelimeler: Dengesizlik, eksik tüketim, Arrow köşesi, indirgenemezlik, sosyal sınıflar.
JEL sınıflandırması: C62 ; D31;

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Since the global economic crisis that started in late 2000s, there has been a growing interest in the Keynesian/Marxian ideas of insufficient demand caused by inequality of income. The Marxian stand can be encapsulated as follows: consumption power of poor workers limits the sales of commodities inducing highly productive resources to remain barren.

Inspired by this view, we develop a simple model with many workers and few capitalists. If low real wages clear the labor market and the productive capacity is high then the gap between the supply of the produced commodities and the consumption of the poor workers cannot be filled by the rich capitalists because the consumption capacity of the rich cannot not be unlimited for all goods. On the other hand, a higher real wage that could support sufficient demand raises labor costs, and induces unemployment. Hence our first result ensues: markets do not equilibrate at any price.

Therefore, under certain conditions, economic activities in our model cannot be carried out in markets that clear. But consumers and firms in real life trade in markets voluntarily subject to some prices, be they market clearing or not. Does the nonexistence of market clearing prices have an effect on voluntary trade? The answer is unequivocal in terms of efficiency: if market clearing prices do not exist then all voluntary allocations at all prices are Pareto-inefficient.[1]Moreover, for any voluntary allocation there is a Pareto-superior allocation that can be supported as a competitive equilibrium with an appropriate redistribution of income. We also prove that these efficiency enhancing redistributions are necessarily income transfers from capitalists to workers.

Nevertheless, the concept of voluntary trade demarcates the set of market allocations by taking the prices as given but does not answer how these prices come about. In order to give an answer consistent with the free markets concept, we assume that prices move flexibly with finite speed according to the imbalances between demand and supply. We prove that these adjusting prices cycle indefinitely if markets do not clear at any price.

What is the relevance of these results to real economies? Our findings suggest that a commonly purported solution to non-clearing markets, to wit, market flexibility might fail as follows. Wages in flexible labor markets would be low inducing inadequate demand for certain goods. However higher wages would give rise to unemployment. But resources would be allocated wastefully in either case. In other words, economic activity in free markets is inefficient when prices cannot maintain market clearing. Moreover, the price adjustment mechanism fails to be self-regulating, and yields only indefinite cycles. Nonetheless, in principle, an elaborate redistribution policy (e.g. welfare state policies) can solve the problem, and promote efficiency by stimulating demand through taxing the rich capitalists and transferring the tax revenues to the poor workers.

The message that we strive to convey is also very similar to the one of Brown and Heal (1979: 573): "It is necessary to consider both the equity and the efficiency dimensions simultaneously." ([2] However, while Brown and Heal invokes non-convexities, our argument is predicated upon the distinction of capitalists and workers. A worker is defined to be a consumer who has only labor to sell in the market whilst capitalists possess also non-labor resources. As Florig (2001a, 2001b) points out "[m]ost of the consumers have only labor to sell." That is, most of the consumers in real life are workers in our parlance. Indeed, existence of many workers and few capitalists is the most salient condition to prove that markets do not clear at any price.

[1] An allocation is voluntary at given prices if no agent benefits from trading less and consumers satisfy their budget constraints. For example, workers selling their labor voluntarily cannot improve their well-beings by working less. We interpret voluntary trade as economic activity in markets without government intervention.

[2] We thank Graciela Chichilnisky for bringing Brown and Heal (1979) to our attention.

Technically, this is equivalent to the nonexistence of competitive equilibrium *à la* Arrow-Debreu.

Chichilnisky (1995: 80) remarks that: "The problem of nonexistence of a competitive equilibrium is pervasive. Despite the fact that market allocations are regarded as a practical solution to the resource allocation problem, many standard economies do not have a competitive equilibrium." Nevertheless, there is a significant paucity of formal analysis in this direction notwithstanding the significance of the subject.

Now let us clarify the novelty and the contributions of this model. First of all, to the best of our knowledge, this is the first study that proves the non-existence of equilibrium for a production economy.[3] The possibility of nonexistence of equilibrium is very-well known due to a famous exchange economy example by Arrow (1952). Yet Arrow's example excludes production, and, therefore, cannot be helpful to see the specific relation between nonexistence of equilibrium and social classes. In particular, all of our results depend on the distinction between workers and capitalists - an observation that is impossible be drawn from Arrow's example.

Second, this is also the first study that analyzes the consequences of the nonexistence of equilibrium. Virtually there is no study that answers how the markets would work if markets do not clear at no price. In this regard, we suggest trade continues on voluntary basis, and prices adjust according the differences between supply and demand. These ensure that all allocations are inefficient and prices cycle indefinitely if there is no competitive equilibrium. Finally, hitherto, that redistribution of income is capable of improving efficiency over all possible voluntary allocations has never been demonstrated.

The remainder of the paper is organized as follows. In Section 2 we develop our model where there is no competitive equilibrium since irreducibility is violated. Efficiency analysis and policy implications are the subjects of Section 3 and 4 respectively. In Section 5 we analyze the Walrasian price adjustment dynamics as a method of determining prices when markets do not clear. The last section concludes with a brief historical discussion of the subject.

The Model

Envisage a town where there are two produced goods: gold and bread. The inputs of gold mining and bread baking are labor and sector specific capital (i.e. machinery). Everyone strictly prefers possessing more gold to less.[4] However, an individual can eat bread up to a certain level of satiation.

To make things concrete, let

$$u_i : X_i \to \mathsf{R}, x \mapsto u_i(x) = \sum_{j=1,2,3} x_j^{\alpha}, \alpha \in (0,1)$$

represent the preferences of individual i such that $X_i \subseteq \mathsf{R}_+^5$. Therefore, there are five goods in the economy. These are gold, bread, labor, mining machinery, and bakery machinery respectively. In other words, good 1 is gold, good 2 is bread, good 3 is labor, and good 4 and 5 are capital goods.[5]

Note that leisure is strictly preferred over working. No one enjoys consuming machinery of any kind.

A consumer is either a capitalist or a worker. The initial endowment of each worker i is

$$e_i = (0,0,1,0,0)$$

[3] Koopmans (1957, p.34-35) gives an example with production but the reason for the non-existence of equilibrium in his example is that consumers cannot survive without trade. However, this condition is irrelevant to our study, and a subject of a very different discussion.

[4] That is to say, gold is a desired good in our example. See Arrow and Debreu (1954, p.280) for a technical definition of desired goods.

[5] For example x_{i1} denotes the consumption of gold by individual 1.

So a worker is a consumer who has to supply labor to buy bread and gold. The initial endowment of each capitalist i is

$$e_i = (0,0,1,m_1,m_2).$$

In contrast to workers, each capitalist owns (m_1,m_2) units of machinery that are specific to mining and bakery. The numbers of the capitalists and the workers are K and W respectively, while $N := W + K$.

It is implausible that a consumer could improve her well-being indefinitely by eating arbitrarily large amounts of bread. Eventually the point of satiation for bread should be relevant. Nor could anyone enjoy leisure more than her own labor-time. Thus we posit, $x \in X_i$ if and only if

$$x \geq 0 \text{ and } x_2 \leq \beta \text{ and } x_3 \leq 1.$$

Here β represents the satiation parameter for bread.[6]

Regarding production, we assume that $Y \subseteq \mathbb{R}^5$ is the following simple fixed coefficient technology: Producing 1 unit of gold and 1 unit of bread requires 2 units of labor, a units of mining machinery and b units of bakery machinery. Formally, $y \in Y$ if and only if

$$y_1 + y_2 + y_3 \leq 0$$
$$ay_1 + y_4 \leq 0$$
$$by_2 + y_5 \leq 0$$

where $(a,b) >> 0$ is a given technology vector. Finally assume that bread baked utilizing all the machinery is enough to satiate the capitalists: $m_2 > b\beta$.

Write θ_i for the profit shares held by individual i. Note that Y is a constant returns to scale technology which makes the distribution of profits irrelevant. By the same token, any amount production is feasible with sufficient labor and machinery. However, inspection reveals that, taking the machinery fixed, production is subject to a full capacity constraint.

The set of all possible prices is $\Delta := \{p \in \mathbb{R}^5_+ : p_1 = 1\}$, i.e. prices are normalized by setting the price of gold to 1. Given $p \in \Delta$, the monetary income of each individual i is

$$m_i(p) := pe_i + \theta_i py$$

The environment is fully characterized by the vector of exogenous

$$\mathsf{E} := (\alpha, \beta, a, b, m_1, m_2, K, W)$$

which we call an economy. Any given economy E satisfies all the standard convexity and closedness conditions, and the preferences are obviously locally non-satiating due to insatiable taste for gold.[7]

The following gives the definition of feasible allocations:

Definition 1 *An allocation* $\xi^* = (x_1^*,...,x_N^*,y^*) \in \mathbb{R}^{5(N+1)}$ *is a* $5-$*tuple of* N *consumption vectors and a production vector;* ξ^* *is feasible iff* $x_i^* \in X_i$ *for all* i, $y^* \in Y$, *and:* $\sum_i x_i^* - e_i \leq y^*$.

Next we define competitive equilibrium.

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[6] We opt to put the satiation parameter in the consumption set. Instead, the idea that bread is subject to satiation could be represented in the rule of the utility function as well. However, the distinction is purely stylistic, and has no material consequences regarding our purposes.

[7] Roughly speaking, a preference relation is locally non-satiating if and only if for any consumption bundle there is another strictly preferred bundle in the close neighborhood of the former bundle. See Moore (2007, p.70) for a formal definition.

Definition 2 *A feasible allocation* ξ^* *is a competitive equilibrium for* E *iff there is a price vector* $p^* \in \Delta$ *such that*

(i) x_i^* maximizes $u_i(x_i)$ subject to $p^* x_i \leq m_i(p^*)$ for each i.

(ii) y^* maximizes $p^* y$ subject to $y \in Y$.

(iii) $p^*\left(\sum_i (x_i^* - e_i) - y^*\right) = 0$.

Our first result states that markets do not clear at any price if the number of workers is high enough.

Theorem 1 *For any* $(\alpha, \beta, a, b, m_1, m_2, K)$ *there is a number* W^* *such that* $W > W^*$ *implies that there is no competitive equilibrium for the economy* E.

All proofs are given in the appendices. However, let us briefly expound the intuition of the above result here. High number of workers implies real wages are low (i.e. "cheap labor" or "poor workers") at least in terms of bread in any competitive equilibrium. This induces two irreconcilable results. According to the first one, cheap labor keeps the supply of bread very close to full capacity. According to the second one, poverty of workers keeps the demand for bread very low. Hence labor market clearing induces a glut in the bread market, which means that market clearing prices do not exist.

Therefore, our first theorem hinges upon the following hypotheses: existence of (i) high number of workers, (ii) full capacity production, and (iii) a consumption good subject to satiation (i.e. bread). We believe these are economically substantive assumptions.

It is noteworthy to underline that non-existence of market clearing prices is of interest independent of the efficiency analysis and policy implications that will be scrutinized in the subsequent sections. Arrow (2005, p.15) states that: "Disequilibria in some markets seem to be clearly observed. I refer to unemployed labor and idle capital equipment." That markets do not clear at any price can explain the ubiquitous disequilibrium in real life that Arrow remarks. However, the explanatory power of this model is highly speculative due to its simple nature and further discussion requires a general formalization.

Voluntary Trade

In this section we ask the following question: if there is no competitive equilibrium how does the economy work? To answer this question we posit that firms and consumers trade voluntarily at given prices.

A feasible allocation is voluntary at given prices if no agent benefits from trading less. For instance, if trade takes place voluntarily then workers cannot increase their utility by selling less labor. In a similar fashion, firms cannot increase profits by producing less.

We maintain that any unfettered market mechanism without government intervention would satisfy voluntariness. Needless to say, competitive equilibrium is a voluntary allocation. Indeed, in economic theory, voluntariness is widely considered as a defining tenet of market mechanisms of any type (e.g. Beviá et al (2003), Svensson (1991), Benassy (1986)). The precise definition is as follows:

Definition 3 *A feasible allocation* ξ^* *is voluntary for* (E, p) *iff: (i)* x_i^* *maximizes* $u_i(x_i)$ *subject to:* $px_i \leq m_i(p)$, *and:*

$$\min\{0, x_{ij}^* - e_{ij}\} \leq x_{ij} - e_{ij} \leq \max\{0, x_{ij}^* - e_{ij}\} \ (j = 2,...,5)$$

(ii) y^* maximizes py subject to: $y \in Y$, and:

$$\min\{0, y_j^*\} \leq y_j \leq \max\{0, y_j^*\} (j = 1,...,5)$$

Note that there is no quantitiy constraint imposed on consumers in the market of good 1. This is a standard practice in the literature and the reason is to impede the trivial alocation (i.e. $x_i = 0$ for all *i*) to qualify as a voluntary allocation.

Now we shall assert that all voluntary allocations at all prices are Pareto-inefficient if there is no competitive equilibrium.

Proposition 1 *Let* ξ *be a voluntary allocation for* E *at prices* $p >> 0$. *If* $W > W^*$ *then* ξ *is Pareto-inefficient.*

What is the intuition of this result? It is easy too see that "price $\geq$ marginal cost" and "marginal rate of substitution $\geq$ relative price" should hold for all goods and consumers in any voluntary allocation if we look closer to the definition of voluntariness.

However, if a voluntary allocation is not perfectly competitive then one of these inequalities should be strict. But when this is the case, as it is widely known, the allocation cannot be Pareto-efficient. In other words, an efficient allocation always equalizes marginal costs and marginal rates of substitution while this is not the case for any voluntary allocation which is not competitive.

Redistribution of Income

Until now, it is demonstrated that any voluntary form of trade is Pareto-inefficient given $W > W^*$. The natural question at this point is whether there is room for some policies to improve over voluntary trade. The answer is affirmative.

Let $q \in \mathsf{R}^{W+K}$ be a redistribution policy such that $\sum q_i = 0$. The monetary income of each consumer is

$$m_i(p) := pe_i + \theta_i py + q_i.$$

Observe that, as an immediate implication of the Second Fundamental Theorem of Welfare, any Pareto-efficient allocation ξ can be supported as a competitive equilibrium with some transfers q. Also recall that, as we proved earlier (see Proposition 1), if there is no competitive equilibrium then any voluntary allocation is Pareto-efficient. Juxtaposing these two results yileds that for any voluntary allocation there is a redistribution policy that supports a Pareto-superior allocation as a competitive equilibrium (if there is no competitive equilibrium without redistribution).

Our next result states that these redistribution policies, that support Pareto-superior allocations as equilibrium, have a specific and salient trait. In particular, these redistribution policies should be

transfers of income from capitalists to workers. Formally,

Theorem 2 *Let* E *be given. Assume* ξ *is a voluntary allocation in this economy. Also write q* for the transfer policy that supports an efficient allocation* ξ^**, which is Pareto-superior to* ξ*, as a competitive equilibrium. If* $W > W^*$ *then* $\sum_{i\in W} q_i^* > 0 > \sum_{i\in K} q_i^*$.

Let us explicate the reason why any redistribution scheme that supports a Pareto-superior allocation should tax the capitalists and transfer the tax revenues to the workers. The economy without taxation cannot achieve a competitive equilibrium due to lack of demand induced by the poverty of workers as we explained before. But, as a response to this problem, if the workers are taxed while the capitalists are transfer receivers then workers become even poorer and hence the lack of demand only worsens and gets deeper. Thus a redistribution policy which aims to surmount the problem we address here should transfer the tax revenues to the workers.

Price Cycles

What determines prices in a free competitive market economy without monopoly pricing and government intervention, if not market clearing? In order to answer this question we invoke the classic Walrasian price adjustment process which determines the path of prices over time.

Suppose that market clearing prices do not exist. Then at all prices there are some over-supplied and over-demanded goods. We assume that prices in those non-clearing markets adjust according to the difference between supply and demand with finite speed.

To formalize this idea define

$$z(p) := \sum_i \left(x_i^* - e_i\right) - y^*$$

where $p \in \Delta$. Here x_i^* and y^* stand for the utility and profit maximizing consumption and production plans at prices p respectively.

Note that the price adjustment process that we suggested above has a solution with $y \neq 0$ only for prices which satisfy $py = 0$ for some $y \in Y$. This is a consequence of profit maximization with a constant returns to scale technology. Indeed, the prices which admit an active production and a well-defined solution to profit maximization are characterized by the following:

$$\begin{aligned} 1 - p_3 - ap_4 &= 0 \\ p_2 - p_3 - bp_5 &= 0. \end{aligned} \tag{1}$$

Hence we stipulate that the price adjustment process satisfies Eq(1).

Now we can present our final result:

Theorem 3 *Assume* $W > W^*$ *for an economy* E. *Then any solution of the differential system*

$$\begin{aligned} \frac{\partial p_j(t)}{\partial t} &= z_j\left(p(t)\right),\ j = 2,3 \\ p(0) &= p_0 \text{ is given} \end{aligned}$$

such that Eq(1) holds at each t is a limit cycle.[8]

In words, if there is no competitive equilibrium then the Walrasian price adjustment process gives rise to indefinite price cycles. Interestingly this result was anticipated by Chipman (1965a) who speculated that prices would oscillate indefinitely if there is no competitive equilibrium.

Of course, our formal result cannot be construed as a dynamic price mechanism for disequilibrium theory but a simple first order approximation. A full-fledged model in this line would be more complicated but also more interesting. Nonetheless, if one interprets these cycles as the probability distribution of prices for a static economy then the result is exact.

Figure 1

The graph of the numerical solution to the Walrasian price adjustment process with $(\alpha,\beta,a,b,m_1,m_2,K,W)=\left(\frac{1}{2},1,1,\frac{1}{2},1,1,1,15\right)$

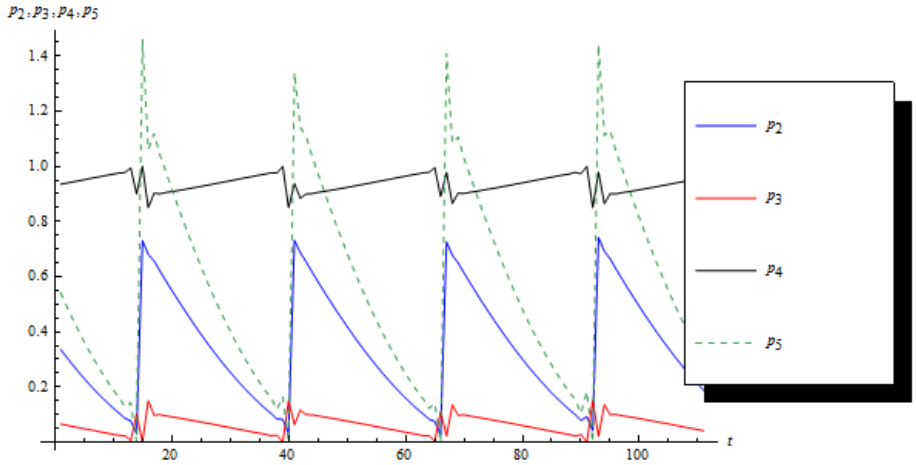

Figure 1 graphs the solution of the Walrasian price adjustment process for a set of parameters which does not permit any Walrasian equilibrium. Typically the relative wage rate with respect to gold has a secular decreasing trend while relative wages with respect to bread has an upward direction. Nevertheless, there are brief intervals (e.g. $t \approx 40$) where prices dramatically veer. During these intervals relative prices enter into a turmoil that strongly resembles market crashes. Soon price adjustment "calms down" until the next "crash" hits. These cycles repeat indefinitely.

Historical Remarks

Explaining why markets do not clear has ancient roots in economic thought. One early attempt in this direction is the Keynesian/Marxian underconsumption theories which are our inspiration.

Underconsumption can be deemed a family of theories which argue that recessions and stagnations occur due to insufficient demand for consumption goods (Bleaney (1976, p.11)). These theories, which are extensively discussed by Bleaney (1976), can be traced back to Malthus and Sismondi, who are prominent proto-Keynesians objecting Say's law. Chipman (1965b, p.707-

[8] Note that the gold market is omitted from the price adjustment process since its price is time invariant due to being normalized to 1.

714) illustrates their objection with an example of competitive equilibrium's nonexistence which is an exchange economy version of our example. This example is due to Arrow (1952, p.527).

However, the particular form of underconsumption that inspired us can be defined as the theory which claims that the reason of capitalist economic crisis is the inability of poor workers to buy back what they produce. This idea, as it is widely known, had a profound impact on the Keynesian tradition.

As Arrow and Hahn (1971, p.347) and Bryant (2010, 1997) point out, the nonexistence of market clearing competitive prices is also germane to the original Keynesian view of markets' incompetency of self-regulation. In particular, that non-clearing market phenomena can be persistent even if there is no price rigidity is a famous theme in the Keynesian tradition. Our analysis can be used to reach this conclusion since even a flexible price mechanism cannot equate demand and supply if there is no market clearing equilibrium.

This relation was noticed during the genesis of modern disequilibrium theory, and early attempts to formalize the Keynesian economics with flexible prices as the nonexistence of market clearing prices can be found in Klein (1947) and Patinkin (1948) even though they lack rigorous micro foundations. Arrow and Debreu (1954, p. 281) give a nonexistence example with proper micro foundations - alas in words - for a production economy. Although their example does not intend to theorize non-clearing market phenomenon, there is a clear reference to disequilibrium and structural unemployment.

Another historical example is Thornton, who contrives examples in which trade takes place at disequilibrium for there is no market clearing equilibrium (see Negishi (1986) for further details). Nevertheless, our result is closer to Malthus and Sismondi's underconsumptionist critique rather than Thornton's approach for Thornton invokes non-convexities which we do not.

The history of the Marxian analysis of underconsumption is beyond the scope of this essay. However, Marx (1967: 316) himself formulates the idea as follows:

> "Contradiction in the capitalist mode of production: the labourers as buyers of commodities are important for the market. But as sellers of their own commodity — labour-power — capitalist society tends to keep them down to the minimum price. Further contradiction: the periods in which capitalist production exerts all its forces regularly turn out to be periods of over-production, because production potentials can never be utilised to such an extent that more value may not only be produced but also realised; but the sale of commodities, ..., is limited, ... by the consumer requirements of a society in which the vast majority are always poor and must always remain poor."

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Appendix

Proof of Theorem 1. Fix a vector $(\alpha, \beta, a, b, m_1, m_2, K) \in \mathsf{R}^7_{++}$. Write

$\mathsf{E}^t := (\alpha, \beta, a, b, m_1, m_2, K, t)$ where t is the number of workers. Now consider a sequence of economies E^t, $t = 1,2,....$ Competitive equilibrium of E^t is denoted by

$$(y^t, x_1^t, ..., x_{K+t}^t, p^t) \in Y \times X_1 \times ... \times X_{K+t} \times \mathsf{R}_+^5.$$

where $p^t = (1, p_2^t, p_3^t, p_4^t, p_5^t)$ is the competitive equilibrium price vector at the t^{th} step. So the price of gold is the numeraire.

Suppose the claim is not true. That is, for all t there exists $r > t$ such that $(y^r, x_1^r, ..., x_{K+t}^r, p^r)$ is nonempty. Now collect all such r's to construct a subsequence such that $(y^r, x_1^r, ..., x_{K+t}^r, p^r)$ is nonempty at each r. Now we shall obtain a contradiction:

Step 1: Neither p_2^r nor p_3^r can be zero at any r. Were $p_3^r = 0$ at some r then no one would supply labor. If there is no labor supply then production of gold and bread should also be zero. That is to say, $y_1^r = y_2^r = y_3^r = 0$. Consequently, the optimal consumption of all individuals should be zero as well. However, the capitalists consume zero only if their income is zero. So $p_4^r = p_5^r = 0$ must be the case in order to ensure that the incomes of the capitalists are zero. In this case, the profit maximization is not well-defined due to constant returns to scale technology for all inputs are free but the price of gold is 1. Contradiction with optimization.

Were $p_2^r = 0$ for some r then bread would be a free good. Thus all consumers would demand bread but the firm would not produce any for $p_3^r > 0$ due to Step 1. Thus, $p_2^r > 0$ for all r.

Step 2: Given p^r, each worker i solves

$$\max_{x_i} \sum_{h=1,2,3} x_{ih}^{\alpha}$$

subject to

$$p_3^r \geq p^r x_i \text{ and } x_i \in R_+ \times [0, \beta] \times [0,1] \times R_+^2.$$

The standard computations show that the solution $x_i^r = (x_{i1}^r, ..., x_{i5}^r)$ satisfies the following property:

$$\text{if } x_{i2}^r < \beta \text{ then } x_{i3}^r = \frac{1}{1 + \left(\frac{p_3^r}{p_2^r}\right)^{\eta} + \left(p_3^r\right)^{\eta}} \tag{2}$$

where $\eta = \alpha/(1-\alpha)$.

Step 3: In this step, we shall prove that $x_{r3}^r \to 1$ as $r \to \infty$. Given p^r, the firm solves

$$\max_{y \in Y} p^r y$$

Let us first see that if y^r is a solution to this optimization problem then $y_1^r + y_2^r + y_3^r = 0$. Were $y_1^r + y_2^r + y_3^r < 0$ then there would be room to decrease labor demand without altering output. However, this implies that y^r cannot be profit maximizing since $p_3^r > 0$ in equilibrium due to Step 1. We deduce $y_1^r + y_2^r + y_3^r = 0$. But

$$ay_1 + y_4 \leq 0 \text{ and } by_2 + y_5 \leq 0$$

give

$$ay_1^r \leq Km_1 \text{ and } by_2^r \leq Km_2$$

since $y_4 + Km_1 \geq 0$ and $y_5 + Km_2 \geq 0$. That is, bread and gold production are curbed by the existing stock of machinery. Juxtaposing $ay_1^r \leq Km_1$ and $ay_2^r \leq Km_2$ and $y_1^r + y_2^r + y_3^r = 0$ implies

$$y_3^r + \lambda \geq 0 \text{ where } \lambda := K\left(\frac{m_1}{a} + \frac{m_2}{b}\right)$$

The interpretation of this result is that λ is an upper bound of labor demand by the firm. But, due to labor market clearing,

$$\sum_{i \in W}\left(1 - x_{i3}^r\right) \leq -y_3^r \leq \lambda$$

where $1 - x_{i3}^r$ is workers i's labor supply. So $x_{r3}^r \to 1$ as $r \to \infty$ for all i.

Step 4: Now we will state a self-evident fact, which ensues due to profit maximization with a fixed coefficient production technology:

$$p_2 - p_3 - bp_5 \leq 0 (\text{with equality if } y_2 > 0).$$

Step 5: The sum of all workers' demand for good 2 (i.e. bread) is $\sum_{i \in W} x_{i2}^r$. Therefore, due to feasibility,

$$\sum_{i \in W} x_{i2}^r \leq K\frac{m_2}{b}.$$

This shows that $x_{r2}^r \to 0$ as $r \to \infty$. However, due to Eq. 2, $x_{r2}^r \to 0$ and $x_{r3}^r \to 1$ imply $\frac{p_3^r}{p_2^r} \to 0$. Thus, applying Step 4, there is r_1 such that $r > r_1$ implies $p_5^r > 0$. But if $p_5^r > 0$ then the supply of bakery machinery is Km_2. This assures that $y_5^r + Km_2 = 0$ due to market clearing. Therefore, $y_2^r = Km_2/b$.

Step 6: Now we shall prove that

$$\sum_{i \in W} x_{i2}^r \leq \frac{p_3^r}{p_2^r}\lambda.$$

Note that

$$x_{i2}^r \leq \frac{p_3^r}{p_2^r}\left(1 - x_{i3}^r\right) \tag{3}$$

holds due to budget constraint of the workers, and $\sum_{i \in W}\left(1 - x_{i3}^r\right) \leq \lambda$ due to Step 3. Therefore,

$$\sum_{i \in W} x_{i2}^r \leq \frac{p_3^r}{p_2^r}\sum_{i \in W}\left(1 - x_{i3}^r\right) \leq \frac{p_3^r}{p_2^r}\lambda.$$

Step 7: Assume $r > r_1$ which ensures $y_2^r = Km_2/b$ due to Step 5. The capitalists' demand for bread is bounded from above by their satiation parameter β. Thus the market clearing condition for bread gives

$$\frac{Km_2}{b} - K\beta - \frac{p_3^r}{p_2^r}\lambda \leq y_2^r - {}_{i=1,\ldots,r+K}\, x_{i2}^r = 0 \tag{4}$$

for all r. On the other hand, there is r_2 such that $r > r_2$ yields

$$\frac{Km_2}{b} - K\beta - \frac{p_3^r}{p_2^r}\lambda > 0$$

since $\frac{p_3^r}{p_2^r} \to 0$ due to Step 5 and $m_2 - b\beta > 0$ by assumption. This inequality contradicts Eq. (4), and competitive equilibrium cannot exist if $r > \max\{r_1, r_2\}$.

Proof of Proposition 1. Suppose not. Then $p>>0$, $W > W^*$, and there is a Pareto-efficient and voluntary allocation ξ. Now we shall obtain a contradiction.

Step 1: This step proves $x_{i1} > 0$ for all i. Note that $pe_i + \theta_i py > 0$ for all i since $p>>0$. However, there is no constraint on good 1 according to the definition of voluntariness. Were $x_{i1} = 0$ for some i then individual i would not be maximizing utility.

Step 2: This step proves $x_{ih} > 0$ for all i and for all $h = 1,2,3$. By hypothesis ξ is Pareto-efficient. Therefore ξ solves the following linear welfare program:

$$\max_{\xi} \sum_i \lambda_i u_i(x_i) \text{ s.t. } \xi \text{ is feasible}$$

for some $\lambda \in \mathsf{R}_+^{W+K}$ such that $\sum_i \lambda_i > 0$.

But $x_{i1} > 0$ due to Step 1. This ensures that $\lambda_i > 0$ for all i. Suppose not. Then there would be an individual i with $\lambda_i = 0$ and another individual j with $\lambda_j > 0$. In this case there exists another allocation where individual i consumes less and individual j consumes more gold (i.e. good 1). Deduce that ξ cannot solve the linear welfare program.

Since $\lambda_i > 0$ for all i, it follows that $x_{ih} > 0$ for all i and for all $h = 1,2,3$. Had there been an individual i and a product h such that $x_{ih} = 0$ then $\partial u_i / \partial x_{ih} = \infty$.

Step 3: This step proves that ξ is a competitive equilibrium. Since $p>>0$, ξ is Pareto-efficient and voluntary, and $x_{ih} > 0$ for all i and for all $h = 1,2,3$, Silvestre (1985, Thereom 2) applies. That is, ξ is a competitive equilibrium.

However, $W > W^*$ by assumption, which implies that there is no competitive equilibrium. Contradiction.

Proof of Theorem 2. For any E the Second Fundamental Theorem of Welfare applies. Therefore, any Pareto-efficient allocation ξ can be supported as a competitive equilibrium with some lump-transfers q.

Fix a vector $(\alpha, \beta, a, b, m_1, m_2, K) \in \mathsf{R}_{++}^7$. Write $\mathsf{E}^t := (\alpha, \beta, a, b, m_1, m_2, K, t) \in \mathsf{R}_{++}^8$ where t is the number of workers. Now consider a sequence of economies E^t, $t = 1,2,\ldots$. Let

$$\xi^t := (y^t, x_1^t, \ldots, x_{K+t}^t) \in Y \times X_1 \times \ldots \times X_{K+t}.$$

be an arbitrary Pareto-efficient allocation. Write $p^t = (1, p_2^t, p_3^t, p_4^t, p_5^t)$ for any competitive equilibrium price vector and q^t for any lump-sum transfers that support the equilibrium at the t^{th} step.

Suppose the claim is not true. That is, for all t there exists $r > t$ such that

$$\sum_{i \in W} q_i^r \le 0 \le \sum_{i \in K} q_i^r$$

for some q^t. Now collect all such r's.

Observe that Step 1-7 of the proof of Theorem 1 apply after making the following changes: replace Eq (2) with

$$\text{if } x_{i2}^r < \beta \text{ then } x_{i3}^r = \frac{1 + \dfrac{q_i^r}{p_3^r}}{1 + \left(\dfrac{p_3^r}{p_2^r}\right)^{\eta} + \left(p_3^r\right)^{\eta}}$$

and Eq (3) with

$$x_{i2}^r \le \frac{p_3^r}{p_2^r}\left(1 - x_{i3}^r\right) + \frac{q_i^r}{p_2^r}.$$

But the conclusion is the nonexistence of competitive equilibrium if $r > \max\{r_1, r_2\}$. Contradiction with q^r supports an equilibrium at each r.

Proof of Theorem 3. Let $p(t)$ solve the differential system. In order to conclude that $p(t)$ is a limit cycle we can apply Poincaré-Bendixson Theorem. We only need to show that $p_j(t),\ j = 2,3$ are bounded. Were $p_3(t)$ unbounded then eventually $1 - p_3 - ap_4 = 0$ would be violated which contradicts that $p(t)$ is a solution. Given that $p_3(t)$ is bounded $p_2(t)$ should also be bounded. Otherwise there would exist a sequence of points in time such that demand for bread eventually shrinks to the consumption demand of the capitalists while $p_2(t)$ increases indefinitely. This yields a negative value to the excess demand function of bread contradicting indefinite increase in $p_2(t)$.